Available online at www.sciencedirect.com

ScienceDirect

International Journal of Heat and Mass Transfer 51 (2008) 3325–3341

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Analysis of the 1D heat conduction problem for a single fin with temperature dependent heat transfer coefficient – Part II. Optimum characteristics of straight plate and cylindrical pin fins

I.N. Dul'kin *, G.I. Garas'ko

All-Russian Electrical Engineering Institute, 12 Krasnokazarmennaya Street, 111250 Moscow, Russia

Received 13 September 2007; received in revised form 22 October 2007
Available online 13 February 2008

Abstract

An exact hypergeometric implicit solution of the 1D steady-state heat conduction problem for a straight fin of constant cross-section is used to calculate the dependence of the main dimensionless fin parameters, specifically, base thermal conductance G and thermo-geometrical fin parameter N on T_e, the ratio of the fin tip to fin base temperature excesses. The straight plate fin (SPF) and cylindrical pin fin (CPF) with an insulated tip (INT) and non-insulated tip (NINT) are optimized. The local heat transfer coefficient (HTC) is assumed to vary as power function of the local fin excess temperature with arbitrary value of exponent n in the range of $-0.5 \leqslant n \leqslant 5$. Every curve from G vs T_e set at given n for a fin with an INT is shown to have a single global maximum $G = G^*_{opt}$ at $T_e = T^*_{e,opt}$ and corresponding $N = N^*_{opt}$, i.e. the main optimum parameters depend only on exponent n. Every curve from G vs T_e set for a fin with a NINT depends, in addition, on the complex fin tip parameter B_ω. These curves have the local maximum and minimum points. As B_ω increases these points approach each other and at $B_\omega = B^{**}$ merge. The corresponding curve G vs T_e has the only inflection point. The main optimum parameters of a fin with an INT and inflection point parameters of a fin with a NINT are approximated by general homographic function of n. Each main optimum parameter of a fin with a NINT is expressed as a product of the corresponding parameter of this fin with an INT and a correction factor approximated by the generalized closed-form formula. The results of the study are presented in form of dimensionless explicit relations, tables and plots which are well suited for the thermal design of optimum fins.

Keywords: Single fin (SPF and CPF); Insulated and non-insulated tip; Temperature dependent HTC; Optimization procedure; Explicit closed-form formulae

1. Introduction

Authors of the fundamental book "Extended Surface Heat Transfer" [1] have pointed out that there are three types of optimizations that pertain to extended surface design and analysis. The first type of optimization involves for longitudinal fins, radial fins, and spines the finding of the profile shape that yields maximum heat flow (or thermal conductance in terms of present study) for a specified mass of a fin (the direct optimization problem) or minimum mass for a specified heat flow or thermal conductance of a fin (the inverse optimization problem). The second type of optimization is, in essence, a variant of the first one for a fin with given shape of profile. For example, optimization of longitudinal fins of constant thickness or the straight plate fins (SPF) and cylindrical spines or cylindrical pin fins (CPF) will be considered in the present paper. The optimization problems of this type are considered in the most publications in available technical literature. The third type of optimization consists in application of the mentioned optimization types to an array of fins in which each fin is operating in an optimum manner.

The paper by Aziz [2] contain a review on optimum dimensions of extended surfaces (mostly, single fins of different shape) losing heat by pure convection to the

* Corresponding author. Tel./fax: +7 495 361 92 26.
E-mail address: heatmass@mtu-net.ru (I.N. Dul'kin).

0017-9310/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2007.11.028

Nomenclature

a, a_e given constants in the heat transfer equation for the lateral surfaces and tip surface of a fin ($\mathrm{W\,m^{-2}\,K^{-(n+1)}}$)
a_p profile area of the SPF ($\mathrm{m^2}$)
A fin cross-sectional area ($\mathrm{m^2}$)
$A_p, \widehat{A}_p$ dimensionless profile area of the SPF (whole and reduced), $A_p = a_p(h_b/k)^2$; $\widehat{A}_p = A_p/2$
B, B_ω general designation for given Biot number or fin base conductance G_z (or G_c) of the SPF (or CPF), $B_\omega = B \cdot f(\omega)$
Bi, Bi_l, Bi_a, Bi_v dimensionless Biot numbers based on the thickness (radius), height, profile area (or volume) of the SPF (or CPF), respectively
E_f extension factor of the total fin heat transfer surface, F/A
$f(\omega)$ power function of the tip heat transfer ratio with different value of p for every given parameter B of the SPF (or CPF), $f(\omega) = \omega^p$
F whole heat transfer area of a fin ($\mathrm{m^2}$)
$C_{i,i=1,3}$ numerical coefficients of the homographic function in Eq. (16)
g_b fin base thermal conductance, Q_b/ϑ_b ($\mathrm{W\,K^{-1}}$)
g_l thermal conductance of a fin with insulated lateral surfaces, kA/l ($\mathrm{W\,K^{-1}}$)
G dimensionless thermal conductance of the SPF (or CPF), $G = \widehat{G}_z/(Bi_a^2/2)^{1/3}$ (or $G = \widehat{G}_c/[Bi_v^3/(4\pi)]^{3/5}$), respectively
G_b dimensionless fin base thermal conductance, g_b/g_l
$G_c, \widehat{G}_c$ dimensionless total and reduced thermal conductance of the CPF, $G_c = g_b(h_b/k^2)$, $\widehat{G}_c = G_c/(4\pi)$
G_{cV} dimensionless thermal conductance of the CPF per unit volume, G_c/V
$G_z, \widehat{G}_z$ dimensionless total and reduced thermal conductance of the SPF, $G_z = g_b/(zk)$; $\widehat{G}_z = G_z/2$
G_{zA_p} dimensionless thermal conductance of the SPF per unit profile area, G_z/A_p
h, h_e heat transfer coefficient for the lateral and tip surfaces of a fin ($\mathrm{W\,m^{-2}\,K^{-2}}$)
K fin augmentation factor (effectiveness), g_b/g_p
k thermal conductivity of the fin material ($\mathrm{W\,m^{-1}\,K^{-1}}$)
l fin height (m)
N dimensionless thermo-geometrical fin parameter, $l\sqrt{h_b P/(kA)}$
n given exponent in the heat transfer equation for a fin
P perimeter of the fin cross-section (m)
r radius of the CPF (m)
Q_{bz}, Q_b heat flow dissipated by the SPF (or CPF), ($\mathrm{W\,m^{-1}}$, (W))
t fin temperature (K)
T dimensionless fin temperature excess, ϑ/ϑ_b
T_e dimensionless fin tip temperature excess, ϑ_e/ϑ_b
v volume of the CPF ($\mathrm{m^3}$)
$V, \widehat{V}$ dimensionless total and reduced volume of the CPF, $V = v(h_b/k)^3$, $\widehat{V} = V/(4\pi)$
x space coordinate (m)
X dimensionless space coordinate, x/l
Y general denotation of the dimensionless dependent variable that have to be optimized
z width of the SPF (m)

Greek symbols
$\beta_{\omega,B}$ dimensionless complex parameter of heat transfer on the fin tip, B_ω/B^{**}
δ fin thickness (m)
ϑ local temperature excess of a fin over the ambient medium, $t - t_a$ (K)
$\xi_{Y,B}$ correction factor to determine the required optimum value, Y_{opt}/Y^*_{opt}
ψ fin aspect ratio (fin height to half-thickness or half-radius ratio), Bi_l/Bi
ω ratio of heat transfer coefficients on the tip and lateral surfaces of a fin, $h_e/h_{x=0}$

Subscripts and superscripts
b, e refer to the fin base and fin tip (for $X = 1$ and $X = 0$, respectively)
opt refer to optimum values
* refer to the fin with an insulated tip
** refer to the inflection point of G vs T_e curve for the fin with a non-insulated tip at $\omega = 1$ and maximum allowable B value (B^{**})

surroundings. The review covers straight (longitudinal) fins, annular (radial) fins and spines of different profile shapes. The optimum dimensions for each shape are given both in terms of the given volume of material as well as in terms of the given heat dissipation. The effects of tip heat loss, variable heat transfer coefficient, internal heat generation, and temperature dependent thermal conductivity of fin material on the optimum dimensions have been discussed.

The direct analytical solution of the optimization problem for a single CPF with an insulated tip and uniform heat transfer coefficient on the cylindrical surface is obtained by Sonn and Bar-Cohen in [3]. Authors have expressed the heat flow Q_b dissipated by a cylindrical spine in terms of its diameter d. The values of fin volume and fin base temperature excess ϑ_b are given. Thermal conductivity of the fin material k and heat transfer coefficient h over the whole fin surface are assumed to be constant. Differentiating Q_b

with respect to d and equating the result to zero authors of [3] have obtained the transcendental equation to determine the dimensionless optimum thermo-geometrical parameter $N^*_{\mathrm{opt}} = 0.919296$ corresponding to maximum point of the curve Q_{b} vs d. Using this N^*_{opt} value the following relations between the required optimum values of d^*_{opt} as well as $Q^*_{\mathrm{b,opt}}$ and given fin parameters l, ϑ_{b}, k and h are obtained: $d^*_{\mathrm{opt}} = 4.73hl^2/k$ and $Q^*_{\mathrm{b,opt}} = 11.736\vartheta_{\mathrm{b}}h^2l^3/k$. Coefficients 4.73 and 11.736 are characteristics of a fin with given profile shape. Such coefficients Razelos in [4] has determined for the circular pin fins of four shapes at different specified parameters and required optimum parameters.

As early as 1979 Razelos has revealed that the main optimum characteristics of a fin with an insulated tip do not depend both on the volume of a fin and the dissipated heat flow (or, in our terms, on the fin base thermal conductance). They depend only on the profile form of a fin. Hereinafter Razelos [4,5] have obtained the explicit dimensionless analytical relations between the required optimum fin parameters and given fin parameter expressed in terms of the main dimensionless fin optimum characteristics N^*_{opt} and D^*_{opt}. The latter parameter is equal to $G^*_{\mathrm{opt}}N^{*1/3}_{\mathrm{opt}}$ for the SPF or $G^*_{\mathrm{opt}}N^{*3/5}_{\mathrm{opt}}$ for the CPF. Taking into account that these parameters are numbers, that characterize a fin of given profile, they have been called profile coefficients.

Optimization problem for the SPF with an insulated or non-insulated tip, with constant, but, in general case, different heat transfer coefficients on the lateral and tip surfaces of a fin ($n = 0, \omega \geqslant 0$) has been considered by Razelos and Krikkis in [6]. In our denotations the dimensionless thermo-geometrical parameter of a fin N and generalized dimensionless parameter $\gamma = \omega\sqrt{Bi}$, that accounts for the heat transfer on the fin tip, were used as independent variables, whereas the dimensionless reduced profile area $\widehat{A}_{\mathrm{p}}$ and thermal conductance at the fin base $\widehat{G}_z$ were used as dependent variables. Taking into account the explicit geometrical and thermal relations between the dependent and independent variables $\widehat{A}_{\mathrm{p}} = N\gamma^3$, $\widehat{G}_z = D\gamma$, where $D = [\tanh(N) + \gamma]/[1 + \gamma\tanh(N)]$, after substituting them into the Euler equation authors [6] have obtained the transcendental equation with respect to N. The root of this equation calculated numerically is the optimum N value denoted N_{opt} for given $\widehat{A}_{\mathrm{p}}$ and ω (or G_z and ω). Parameters N_{opt}, D_{opt} as functions of complex parameters $\widehat{A}_{\mathrm{p},\omega} = \widehat{A}_{\mathrm{p}}\omega^3$ and $G_{z,\omega} = G_z\omega$ have been calculated numerically and approximated by the polynomial expressions.

The remarkable study relating to analysis and optimum design of the straight plate fins (SPF) and cylindrical pin fins (CPF) with variable heat transfer coefficient has been presented in paper by Chung and Iyer [7]. The heat balance integral approach has been extended in this paper to determine the optimum dimensions of the SPF and CPF by incorporating transverse heat conduction. Heat transfer coefficient was assumed to be power functions not only of ϑ but also of characteristic fin length with different values of exponents. The effect of the heat transfer from the fin tip is also taken into account. The obtained mathematical expression for optimum fin height to thickness ratio is found to be as simple as that of classical one-dimensional approach, but the accuracy of numerical results for the low values of this ratio (in the range of 1–4) has been improved significantly.

Yeh and Liaw [8] have obtained an exact solution for thermal characteristics of fins with power-law heat transfer coefficient. Yeh [9] has investigated analytically the optimum dimensions of the SPF and CPF considering temperature dependent heat transfer coefficient and heat transfer from the fin tip. The fin volume is fixed to obtain the aspect ratios of the uniform area cross-section fins with maximum heat transfer rates. The characteristic length has been taken into consideration in heat transfer coefficient in addition to the local excess temperature. The analysis has shown that an optimum aspect ratio does not exist for the fins with heat transfer from the tip if $Bi_{\mathrm{a}} > Bi^{**}_{\mathrm{a}}$ or $Bi_{\mathrm{v}} > Bi^{**}_{\mathrm{v}}$ (in denotations of the present paper). However, there always exists an optimum aspect ratio for an insulated-tip fin. The optimum aspect ratio of a fin is highest for a fin with an insulated tip and decreases with increasing rate of heat transfer from the tip. Yeh [10] has investigated also the errors in the one-dimensional fin optimization solution for convective heat transfer by comparison it with modified one-dimensional and two-dimensional solutions. The latter solution has been computed numerically by the iteration method. All solutions were performed for the SPF and CPF with constant but different heat transfer coefficients on the lateral surfaces of a fin and on the fin tip. It was found that for fins with an insulated tip optimization is possible for any value of Biot number Bi_{a} and Bi_{v}. For fins with a non-insulated tip there exist maximum value of Biot number $Bi^{**}_{\mathrm{a}} = 0.18$ for the SPF and $Bi^{**}_{\mathrm{v}} = 0.055$ for the CPF over which the optimum fin dimensions do not exist. One-dimensional approximation is valid with high accuracy for $Bi_{\mathrm{a}} \leqslant 0.08$ and $Bi_{\mathrm{v}} \leqslant 0.02$.

The purpose of the present study is to develop the unified approach to determine the any required optimum characteristic of the SPF and CPF with an insulated or non-insulated tip subjected to power-law relation between the local heat transfer coefficient and corresponding fin temperature excess. This approach is based on the exact implicit hypergeometric solution of the 1D heat conduction problem for a fin in terms of G vs T_{e} and N vs T_{e} relations at given value of n for a fin with an insulated tip (INT) and, in addition, of dimensionless complex fin tip parameter $\omega^{\mathrm{p}}B$ for a fin with a non-insulated tip (NINT). These relations were calculated in the range of $0 \leqslant T_{\mathrm{e}} \leqslant 1$, $-0.5 \leqslant n \leqslant 5$, and $0 \leqslant B \leqslant B^{**}$ at $\omega = 1$. The values of T_{e} and N corresponding to global or local maximum of G vs T_{e} relation at every given n for a fin with an INT or n and B for a fin with a NINT are determined and denoted as the main optimum fin parameters $T_{\mathrm{e,opt}}$, N_{opt} and G_{opt}.

Main optimum parameters of a fin with an INT and maximum allowable fin parameters B are approximated by the general explicit closed-form function of exponent n. The plots and analysis considered below show that the main optimum parameter of a fin with a NINT can be presented as a product of the corresponding optimum parameter of this fin with an INT, depending only on n, and a correction factor, depending on n and, in addition, on given complex fin tip parameter $B_\omega = \omega^p B$. Thus, the purpose of the present paper is to obtain the explicit closed-form relations to determine any optimum parameter of the SPF and CPF with an INT and a NINT for a direct and inverse optimization problems.

2. Analysis of the fin optimization problem

2.1. Formulation of the fin optimization problem

Two types of single fins are considered for the optimization, specifically a longitudinal straight fin with uniform thickness (straight plate fin) designated below as SPF, and a cylindrical spine, or cylindrical pin fin, designated as CPF. We look at the optimization problem within the framework of a unified approach, both for direct and inverse problems. The first specified value for both forms of the optimization problem is an exponent n in the power-law type relation between the local heat transfer coefficient and the excess temperature of the fin.

The direct statement of the optimization problem for a single fin can be formulated as follows: it is required to find the geometrical dimensions of a fin, specifically, the dimensionless half-thickness of the SPF $Bi \equiv h_b\delta/(2k)$ (or the half-radius of the CPF $Bi \equiv h_b r/(2k)$) and the dimensionless height of a fin $Bi_l \equiv h_b l/k$ needed to provide for the maximum overall thermal conductance of the SPF $G_z = g_b/(kz)$ or CPF $G_c = g_b h_b/k^2$. The given values here are

– For the SPF, the exponent n and the dimensionless profile area $A_p = a_p(h_b/k)^2$, or corresponding Biot number $Bi_a = A_p^{1/2} = a_p^{1/2}(h_b/k)$ instead of the fin profile area.
– For the CPF, the exponent n and the dimensionless volume $V = v(h_b/k)^3$, or the corresponding Biot number $Bi_v = V^{1/3} = v^{1/3}(h_b/k)$ instead of the fin volume.

To determine these optimum dimensions we have to find the peak points of the functions G vs T_e for the given values of n and Bi_a or Bi_v, and the points on the curves N vs T_e that correspond to these peaks, for the same values of n and Bi_a or Bi_v. The values corresponding to these peak points of the curves G vs T_e will be named optimum and denoted below by subscript "opt". With knowledge of N_{opt} and G_{opt}, any one of optimum geometrical and thermal parameters of the fin can be determined by means of Tables 1 and 2 in the first part of this study [11]. Using the relations collected in these tables we can solve the direct optimization problem even when either Bi_a or Bi_v of the fin is not given as the second independent variable, but one of its geometrical dimensions, specifically, half-thickness of the SPF (half-radius of the CPF) Bi or height of the fin Bi_l is given. Besides the geometrical parameters mentioned above, the optimum fin conductance $G_{z,opt}$ or $G_{c,opt}$ and specific thermal conductance $G_{zA_p,opt} \equiv G_{z,opt}/A_p$ (or $G_{cV,opt} \equiv G_{c,opt}/V$) can be found for the SPF and CPF, respectively, as well as fin augmentation factor (effectiveness) K_{opt} and fin efficiency η_{opt}.

The given values for the inverse formulation of the fin optimization problem are the exponent n in Eq. (1) and the fin thermal conductance $G_z \equiv g_b/(kz)$ for the SPF or $G_c \equiv g_b(h_b/k^2)$ for the CPF.

The required values which we need to determine in this case are the minimum profile area of the SPF $A_{p,opt}$ (or corresponding Biot number $Bi_{a,opt} = A_{p,opt}^{1/2}$) or the minimum volume of the CPF V_{opt} (or the corresponding Biot number $Bi_{v,opt} = V_{opt}^{1/3}$), as well as other optimum characteristics mentioned above, to provide for the given fin base thermal conductance at the given value of n.

2.2. Formulae to calculate G vs T_e and N vs T_e relations and determine the main optimum parameters of the SPF and CPF

The basic relations for the SPF and CPF with a non-insulated tip in general case of arbitrary value of n have been considered in the first part of this study. The next inte-

Table 1
Coefficients C_i of the homographic function Eq. (16) intended for approximation of the main optimum parameters Y^*_{opt} vs n and Y^{**}_{opt} vs n of the SPF and CPF with insulated and non-insulated tips

Fin type	Optimum parameters Y^*_{opt}, Y^{**}_{opt}	Coefficient C_i for i		
		1	2	3
SPF	N^*_{opt}	1.41922	0.05912	0.24345
	G^*_{opt}	0.79146	0.14739	0.44037
	$T^*_{e,opt}$	0.45706	0.53484	0.52842
CPF	N^*_{opt}	0.9193	0.11223	0.48075
	G^*_{opt}	0.76314	0.20122	0.44145
	$T^*_{e,opt}$	0.68815	0.74102	0.74009
SPF	N^{**}_{opt}	0.71421	0.0513	0.29448
	G^{**}_{opt}	0.85462	0.17243	0.47289
	$T^{**}_{e,opt}$	0.67289	0.59597	0.59569
CPF	N^{**}_{opt}	0.46944	0.06275	0.52651
	G^{**}_{opt}	0.83683	0.23356	0.46613
	$T^{**}_{e,opt}$	0.85340	0.80322	0.80282

Table 2
Coefficients C_i of the homographic function Eq. (16) intended for approximation of the maximum allowable values of B^{**} and exponents p in Eqs. (18) and (19) for the SPF and CPF with non-insulated tips

Fin type	Maximum allowable parameter B^{**}	Coefficient C_i for i			p
		1	2	3	
SPF	Bi^{**}_a	0.1803	0	0.69236	3/2
	Bi^{**}	0.0803	0	0.7758	2
	Bi^{**}_l	0.2024	0	0.60811	1
	G^{**}_z	0.43297	0	0.72642	1
CPF	Bi^{**}_v	0.05446	0	0.8745	5/3
	Bi^{**}	0.01496	0	0.89576	2
	Bi^{**}_l	0.05742	0	0.83131	1
	G^{**}_c	0.01295	−0.00204	1.25517	3

gral relation between the thermo-geometrical parameter N of a fin with a non-insulated tip and the dimensionless fin tip temperature T_e is valid for the straight fins and spines with any form of constant cross-section. As it follows from Eq. (6) [11]:

for $n \neq -2$

$$N = \int_{T_e}^{1} dT / \sqrt{[2/(n+2)]\{T^{n+2} - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}. \tag{1}$$

In contrast to, the generalized dimensionless thermal conductance of these fins depends on the shape of fin cross-section. According to Eqs. (59) and (62) for the SPF [11]:

$$G \equiv \frac{\widehat{G}_z}{\widehat{A}_p^{1/3}} = \frac{\sqrt{[2/(n+2)]\{1 - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}}{N^{1/3}} \tag{2}$$

for the CPF

$$G \equiv \frac{\widehat{G}_c}{\widehat{V}^{3/5}} = \frac{\sqrt{[2/(n+2)]\{1 - T_e^{n+2}[1-(n+2)\omega^2 Bi T_e^n/2]\}}}{N^{3/5}}, \tag{3}$$

where $\omega = h_e/h_{x=0} = a_e/a$ is the ratio of the heat transfer coefficients from the tip surface and lateral surfaces of a fin at the tip coordinate $x = 0$, $Bi = h_b A/(kP)$ is the transverse Biot number of a fin, and $\gamma^2 = \omega^2 Bi$ is the same parameter as in Eq. (1). It will be considered below as a single parameter.

The relation between N and T_e in Eq. (1) is conveniently to express using the exact hypergeometric formulae [8,9].

for $n \geqslant 0$

$$N = \sqrt{\frac{2}{(n+2)} \frac{\chi}{T_e^n}} F\left(\frac{1}{2}; \frac{n+4}{2(n+2)}, \frac{3}{2}, \chi\right) - \omega\sqrt{Bi} F\left(\frac{1}{2}; \frac{n+4}{2(n+2)}, \frac{3}{2}, \beta\right) \tag{4}$$

for $-2 < n < 0$

$$N = \sqrt{\frac{2}{(n+2)} \chi} F\left(1; \frac{n+1}{n+2}, \frac{3}{2}, \chi\right) - \omega\sqrt{Bi} F\left(1; \frac{n+1}{n+2}, \frac{3}{2}, \beta\right), \tag{5}$$

where

$$\beta = \frac{(n+2)}{2} \omega^2 Bi T_e^n, \tag{6}$$

$$\chi = 1 - T_e^{(n+2)}(1-\beta) \tag{7}$$

and the three-parametric hypergeometric functions are denoted by symbol F.

The thermal conductances of the SPF and CPF expressed by Eqs. (2) and (3) can be written in unified form as

$$G = \frac{1}{N^r} \sqrt{\frac{2}{n+2} \chi}, \tag{8}$$

where $r = 1/3$ for the SPF and $r = 3/5$ for the CPF.

To express Bi and $\sqrt{Bi}$ in Eqs. (4)–(6) in terms of given parameters Bi_a or Bi_v using Tables 1 and 2 in the first part of this study [11], to multiply these values by ω^2 and ω, designate $Bi_{a,\omega} = \omega^{3/2} Bi_a$, $Bi_{v,\omega} = \omega^{5/3} Bi_v$ for the SPF and CPF, respectively, we obtain

for the SPF

$$\omega^2 Bi = [Bi_{a,\omega}^2/(2N)]^{2/3}, \quad \omega\sqrt{Bi} = [Bi_{a,\omega}^2/(2N)]^{1/3} \tag{9}$$

for the CPF

$$\omega^2 Bi = [Bi_{v,\omega}^3/(4\pi N)]^{2/5}, \quad \omega\sqrt{Bi} = [Bi_{v,\omega}^3/(4\pi N)]^{1/5}. \tag{10}$$

To find with a high accuracy the main optimum values $T_{e,opt}$ and N_{opt} corresponding to the local maximum G_{opt} of the curve G vs T_e at given parameters $-0.5 \leqslant n \leqslant 5$ and $0 \leqslant Bi_{a,\omega} \leqslant Bi_a^{**}$ (or $0 \leqslant Bi_{v,\omega} \leqslant Bi_v^{**}$) we use the following method. First, we take the total derivative of G with respect to T_e in Eq. (8) taking into consideration Eqs. (6) and (7)

$$\frac{dG}{dT_e} = \frac{\partial G}{\partial T_e} + \frac{\partial G}{\partial N}\frac{dN}{dT_e} = 0. \tag{11}$$

Second, the implicit with respect to N Eqs. (4) or (5) after substitution of Eqs. (6), (7) and (9), (10) may be written as follows:

$$f(N, T_e) = N - I(N, T_e) = 0, \tag{12}$$

where $I(N, T_e)$ is the RHS of Eqs. (4) or (5). Differentiating Eq. (12) with respect to T_e, gives

$$\frac{df}{dT_e} = \frac{\partial f}{\partial T_e} + \frac{\partial f}{\partial N}\frac{dN}{dT_e} = 0, \tag{13}$$

whence it follows that

$$\frac{dN}{dT_e} = -\frac{\partial f}{\partial T_e} \Big/ \frac{\partial f}{\partial N}. \tag{14}$$

Substitution of Eq. (14) into Eq. (11) gives the Euler equation

$$\left(\frac{\partial G}{\partial T_e}\right)\left(\frac{\partial f}{\partial N}\right) - \left(\frac{\partial G}{\partial N}\right)\left(\frac{\partial f}{\partial T_e}\right) = 0. \tag{15}$$

The direct calculation of the relations N vs T_e and G vs T_e is performed numerically with help of Maple package using Eqs. (4)–(6) and taking into account for Eqs. (7)–(10) at variation of T_e with a small step $\Delta T_e = 0.001$ in the range of $0 \leqslant T_e \leqslant 1$ for the given values of n in the range $-0.5 \leqslant n \leqslant 5$ at $\omega = 1$. It can be seen from Eqs. (1)–(6) that parameters ω and Bi_a (or Bi_v) appear as a single complex parameter $Bi_{a,\omega} = \omega^{3/2} Bi_a$ for the SPF (or $Bi_{v,\omega} = \omega^{5/3} Bi_v$ for the CPF). Therefore, results of these calculations performed only for $\omega = 1$ can be extended to any ω

values that satisfy to the relations $0 \leqslant \omega^{3/2}Bi_a \leqslant Bi_a^{**}$ for the SPF and $0 \leqslant \omega^{5/3}Bi_v \leqslant Bi_v^{**}$ for the CPF. For a fin with an insulated tip parameter ω in Eqs. (1)–(6) is equal to zero. All main fin characteristics in this case depend only on n and will be denoted below by the superscript "$*$" (asterisk).

A search for the optimum values is performed numerically by simultaneous solution of Eqs. (11) and (15) in view of Eqs. (4)–(10) using Maple package for $-0.5 \leqslant n \leqslant 5$, $0 \leqslant Bi_a \leqslant Bi_a^{**}$ or $0 \leqslant Bi_v \leqslant Bi_v^{**}$ at $\omega = 1$.

3. Results and discussion

3.1. G vs T_e and N vs T_e relations and optimum fin parameters

The calculation results are shown in Fig. 1a–d for the SPF and CPF, respectively. Two sets of curves G vs T_e and N vs T_e are presented on a single plot for every n and fin shape. As an example, these plots are given for fins with uniform heat transfer coefficient over the whole fin surface ($n = 0$) and for nucleate boiling on the fin surface ($n = 2$). Plots for other values of n have a similar view. The curve 1 corresponds to a fin with an insulated tip. In this case parameter ω in the RHS of Eqs. (4)–(6) is equal to zero. The curves 2–6 in Fig. 1a and b correspond to the SPF with non-insulated tips at $Bi_{a,\omega}$ in the ranges 0.02–0.3 and 0.02–0.1 for $n = 0$ and $n = 2$, respectively. The curves 2–6 in Fig. 1c and d correspond to the CPF with non-insulated tips at $Bi_{v,\omega}$ in the ranges 0.01–0.1 and 0.002–0.04 for $n = 0$ and $n = 2$, respectively.

It is seen that the curve 1 and curves 2–6 from the G vs T_e set have the same character at the left side of T_e range and differ principally at the right side of the range. The value of G is equal to zero for the infinitely high fins ($N \to \infty$) corresponding to $T_e = 0$, for any given value of

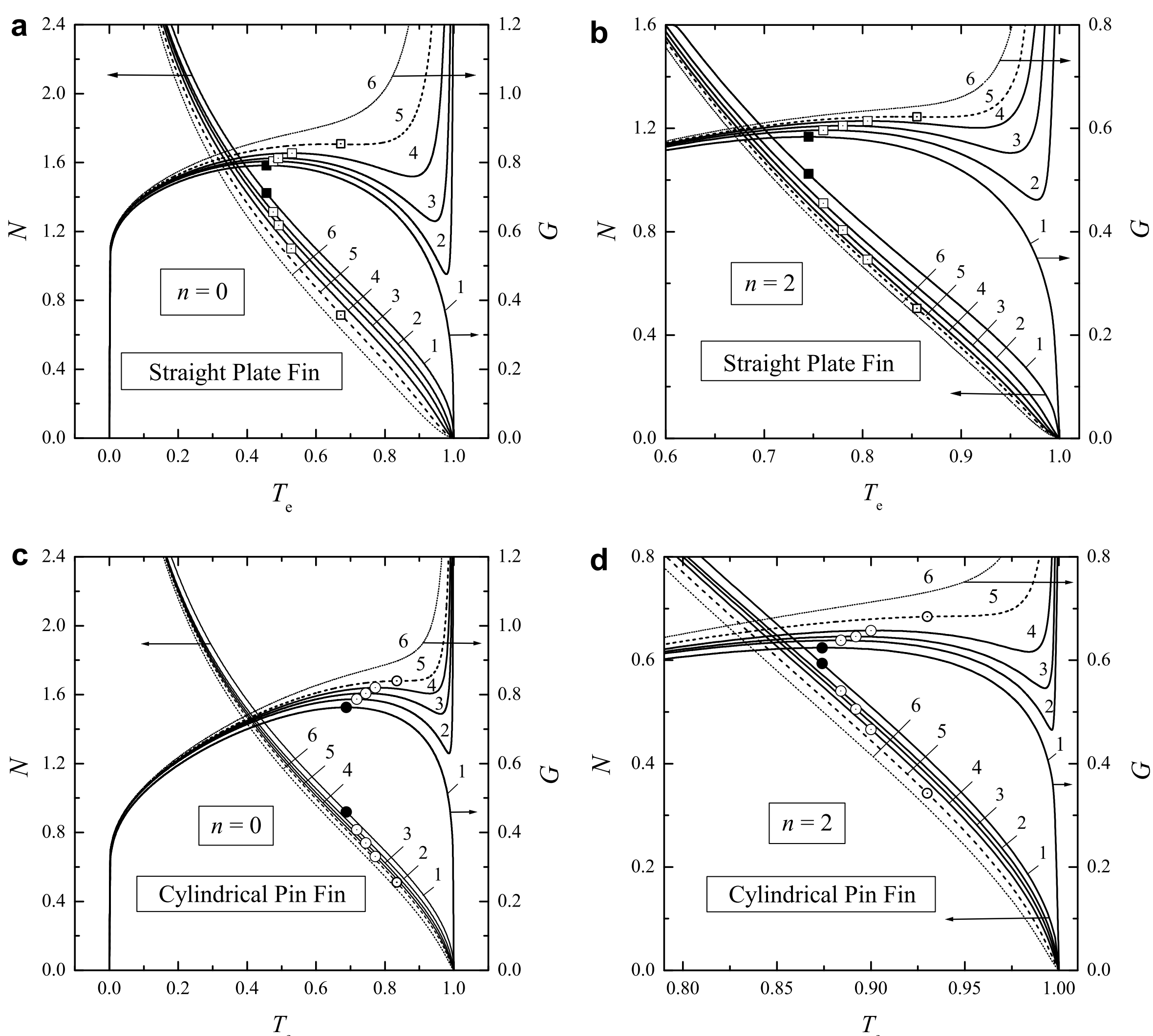

Fig. 1. Predicted set of curves G vs T_e (bottom and top arrows pointing to right ordinate axis) and corresponding set of curves N vs T_e (arrow pointing to left ordinate axis) for the SPF (a and b) and CPF (c and *d*) at $n = 0$ and $n = 2$. Curves 1 refer to fins with INT($\omega = 0$ at any given value of Bi_a or Bi_v), curves 2–6 refer to fins with NINT ($\omega = 1$ at $Bi_a = 0.02, 0.05, 0.1, 0.1803, 0.3$ and $Bi_v = 0.01, 0.025, 0.04, 0.0545, 0.1$ for $n = 0$ as well as $Bi_a = 0.02, 0.04, 0.06, 0.0743, 0.1$ and $Bi_v = 0.002, 0.005, 0.01, 0.0197, 0.04$ for $n = 2$). Solid and dot-centered open squares and circles give the optimum values of the main dimensionless parameters of the SPF and CPF, respectively.

$Bi_{a,\omega}$ or $Bi_{v,\omega}$, since in this case the transverse fin dimension Bi is equal to zero. With increase of T_e, the values of G also increase initially very sharply at T_e near $T_e = 0$ and then slowly. The curve 1 for a fin with an insulated tip ($\omega = 0$) reaches the single peak value $G = G^*_{opt}$ at $T_e = T^*_{e,opt}$ (solid squares and circles) and then decreases up to zero at $T_e = 1$, that corresponds to zero thermal conductance of the insulated fin tip. The point on the curve 1 from the N vs T_e set corresponding to the peak point on the curve 1 from the G vs T_e set (at the same value of $T^*_{e,opt}$), which gives the optimum parameter of a fin with an insulated tip $N = N^*_{opt}$ which is also denoted by a solid square or circle. Hence, the optimum values of G^*_{opt}, $T^*_{e,opt}$, and N^*_{opt} for a fin of given form (SPF or CPF) with an insulated tip depend only on exponent n in the heat transfer equation (1).

For fins with non-insulated tips curves 2–4 also increase with the increase of T_e, reach the local maxima $G = G_{opt}$ at $T_e = T_{e,opt}$ denoted by a dot-centered open squares or circles, and then decrease again up to some local minimum value. The abscissa $T_{e,min}$ and ordinate G_{min} of this minimum depends on $Bi_{a,\omega}$ or $Bi_{v,\omega}$ for the given value of n. The greater is the $Bi_{a,\omega}$ or $Bi_{v,\omega}$, the lesser is the $T_{e,min}$ and the greater is the G_{min}. Once $Bi_{a,\omega}$ or $Bi_{v,\omega}$ reach such value that G_{min} becomes equal to G_{opt} the corresponding short-dashed curve 5 from the G vs T_e set has the single inflection point where two conditions are valid simultaneously $dG/dT_e = 0$ and $d^2G/dT_e^2 = 0$. We assume that this value of $Bi_{a,\omega}$ and $Bi_{v,\omega}$ is the maximum possible allowable value for the physically meaningful optimization problem and denote it Bi^{**}_a and Bi^{**}_v.

Since the curves displayed in Fig. 1 by short-dotted lines 6 at $Bi_{a,\omega} > Bi^{**}_a$ and $Bi_{v,\omega} > Bi^{**}_v$ have no maximum values, the physical meaning of the optimization problem under these conditions is lost.

As $T_e > T_{e,min}$ for curves 2–4, the parameter G increases again, and tends to infinity at $T_e = 1$ as the height of the fin approaches zero. This means that a fin with a given value of $Bi_{a,\omega}$ or $Bi_{v,\omega}$ vanishes (i.e. as if it is "spread" on the base wall with infinite area). The corresponding N values from N vs T_e set approach zero when T_e approaches 1 for any value of $Bi_{a,\omega}$ or $Bi_{v,\omega}$.

The calculation results expressed by the function G vs T_e are presented in Fig. 2a and b, respectively, for the SPF and CPF with an insulated ($\omega = 0$) and non-insulated tip ($\omega = 1$). In the latter case the results are presented only for maximum allowable Biot numbers $Bi_a = Bi^{**}_a$ and $Bi_v = Bi^{**}_v$ (i.e. $Bi_{a,\omega} \equiv \omega^p Bi_a = Bi^{**}_a$ and $Bi_{v,\omega} \equiv \omega^p Bi_v = Bi^{**}_v$) at different values of n. Corresponding plots N vs T_e are presented in Fig. 3a and b. For fins with insulated tips the curves are displayed by solid lines. Maxima of the functions G vs T_e and the corresponding points on N vs T_e curves are denoted by solid squares and circles. The curves G vs T_e and N vs T_e for fins with non-insulated tips at $Bi_a = Bi^{**}_a$, $Bi_v = Bi^{**}_v$ and different values of n are displayed by short-dashed lines. The deflection points of the curves G vs T_e and the corresponding points on N vs T_e curves are denoted by dot-centered open squares and circles for the SPF and CPF, respectively.

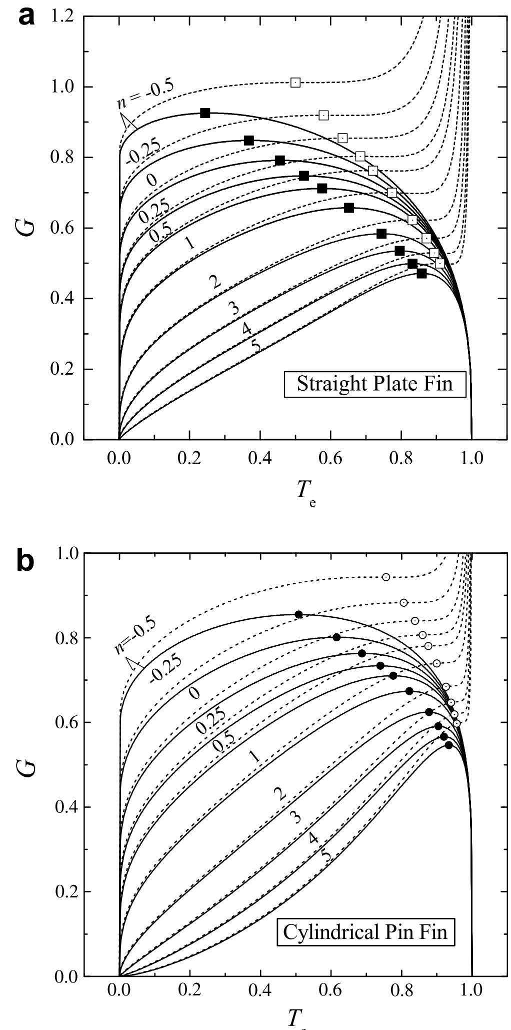

Fig. 2. Predicted set of curves G vs T_e for the SPF (a) and CPF (b). Solid curves refer to fins with INT ($\omega = 0$ at any given value of Bi_a or Bi_v). Short-dashed curves refer to fins with NINT at $Bi_a = Bi^{**}_a$ and $Bi_v = Bi^{**}_v$ for different values of n. Solid squares and circles are the maximum points of the solid curves, dot-centered open squares and circles are the inflection points of the corresponding short-dashed curves for the same values of n.

With reference to Figs. 2 and 3 it can be seen that if n increases, so both sets of curves G vs T_e (solid and dashed) pass lower whereas the corresponding sets of curves N vs T_e pass righter. In these cases, G^*_{opt} and G^{**}_{opt} as well as N^*_{opt} and N^{**}_{opt} values decrease. For example, when the parameter n increases from -0.5 to 5, the values of G^*_{opt} and G^{**}_{opt} as well as N^*_{opt} and N^{**}_{opt} are reduced by about half. The corresponding values of $T^*_{e,opt}$ increase by the factor 3.3 whereas values of $T^{**}_{e,opt}$ increase only by the factor 1.82.

Eliminating parameter T_e from the relations G vs T_e and N vs T_e shown in Figs. 2 and 3, we obtain the relation G vs N displayed in Fig. 4a and b for the SPF and CPF with an insulated and non-insulated tip.

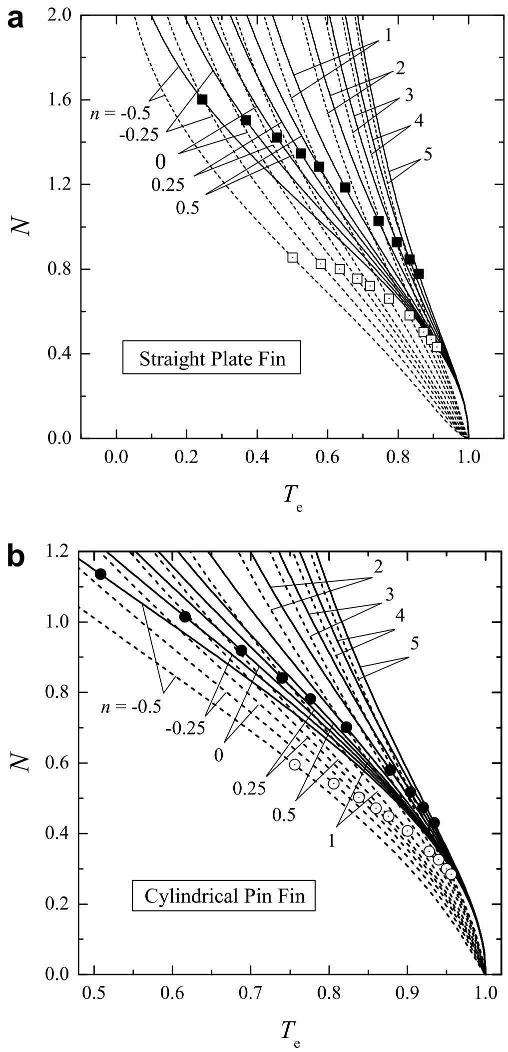

Fig. 3. Predicted set of curves N vs T_e for the SPF (a) and CPF (b). Solid curves refer to fins with INT ($\omega = 0$ at any given value of Bi_a or Bi_v). Short-dashed curves refer to fins with NINT at $Bi_a = Bi_a^{**}$ and $Bi_v = Bi_v^{**}$ for different values of n. Solid and dot-centered open squares and circles correspond to the maximum and inflection points of the solid and short-dashed curves sets G vs T_e in Fig. 2 (a and b) for the same values of n.

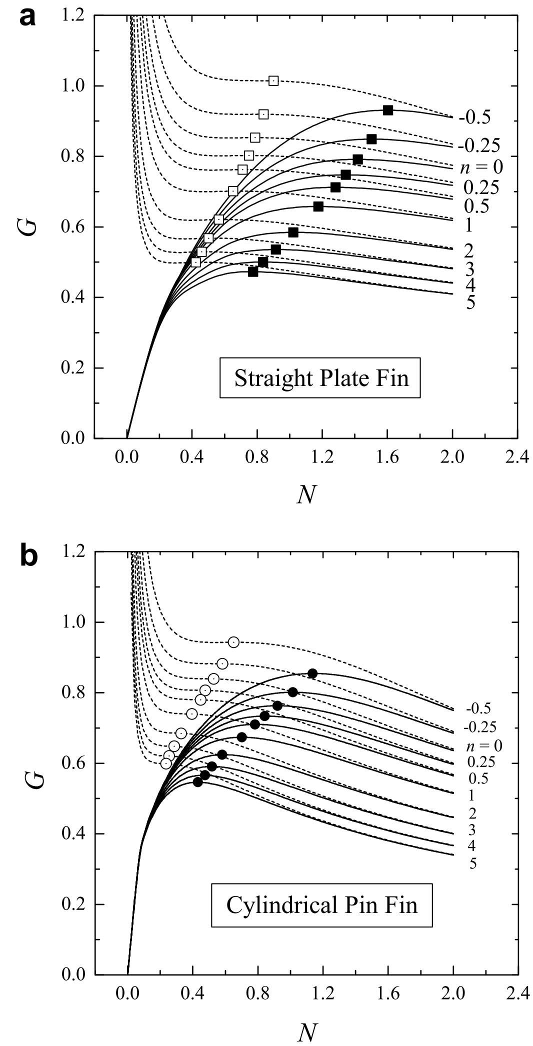

Fig. 4. Predicted set of curves G vs N for the SPF (a) and CPF (b). Solid curves refer to fins with INT ($\omega = 0$ at any given value of Bi_a or Bi_v). Short-dashed curves refer to fins with NINT at $Bi_a = Bi_a^{**}$ and $Bi_v = Bi_v^{**}$ for different values of n. Solid squares and circles are the maximum points of the solid curves, dot-centered open squares and circles are the inflection points of the corresponding short-dashed curves for the same values of exponent n.

The comparison of the optimum values for fins with insulated and non-insulated tips shows that $G_{opt} > G^*_{opt}$ and $T_{e,opt} > T^*_{e,opt}$, whereas $N_{opt} < N^*_{opt}$. The value of G^{**}_{opt} is greater than G^*_{opt} for the n values as low as 6–9.4% (the first value is valid for $n = 5$ and the second for $n = -0.5$). The value of $T^{**}_{e,opt}$ is greater than $T^*_{e,opt}$ about by half for $n = -0.5$ and as low as 6% for $n = 5$. The value of N^{**}_{opt} are lesser than the corresponding N^*_{opt} value by the factor of about 1.8 for all n values. The difference between the main optimum parameters for fins with non-insulated and insulated tips is greater for the CPF than for the SPF. In addition, the optimal parameters for fins with non-insulated tips depend not only on n, but also on the given parameters ω and Bi_a or Bi_v. However, these parameters appear only in form of a single complex parameter $Bi_{a,\omega} = \omega^{3/2} Bi_a$ or $Bi_{v,\omega} = \omega^{5/3} Bi_v$.

3.2. Approximate expression to determine the optimum parameters for a fin with an insulated tip and maximum allowable given parameters B^{**} for a fin with a non-insulated tip

Let us designate:

(1) Main required optimum characteristics of a fin with an insulated tip G^*_{opt}, $T^*_{e,opt}$ and N^*_{opt} by the general denotation Y^*_{opt}.
(2) Parameters G^{**}_{opt}, $T^{**}_{e,opt}, N^{**}_{opt}$ corresponding to the inflection point for maximum allowable value of Bi^{**}_a or Bi^{**}_v by the general denotation Y^{**}_{opt}.
(3) Either from the maximum allowable Biot numbers (Bi^{**}_a or Bi^{**}_v, Bi^{**} and Bi^{**}_l) for the direct statement of the fin optimization problem or the thermal conduc-

tance of the SPF or CPF (G_z or G_c) for the inverse statement of this problem by the general denotation B^{**}.

Parameters Y^*_{opt}, Y^{**}_{opt} and B^{**} for a fin of given form depend only on exponent n in the equation for the local heat transfer coefficient from the fin surface. Results of numerical calculations are approximated by the general for all these parameters homographic function with three numerical coefficients:

$$Y^*_{opt} \text{ or } Y^{**}_{opt} \text{ or } B^{**} = \frac{C_1 + C_2 n}{1 + C_3 n}. \qquad (16)$$

Coefficients C_1–C_3 in Eq. (16) to calculate Y^*_{opt} and Y^{**}_{opt} for the SPF and CPF are collected in Table 1, as well as the corresponding coefficients to calculate B^{**} are given in Table 2.

The approximate curves N^*_{opt}, G^*_{opt}, and $T^*_{e,opt}$ vs n as well as N^{**}_{opt}, G^{**}_{opt}, and $T^{**}_{e,opt}$ vs n determined using Eq. (16) and displayed in Fig. 5a–c by solid and short-dashed lines are in good agreement with corresponding values predicted numerically and shown by solid and dot-centered open squares and circles, respectively, at different values of n.

The functions B^{**} vs n i.e. Bi^{**}_a and Bi^{**}_v, as well as Bi^{**}, Bi^{**}_l, G^{**}_z and G^{**}_c are shown in Fig. 6a and b for the SPF and CPF, respectively. The predicted data in Fig. 6 approximated by Eq. (16) are displayed by short-dashed lines. Accuracy of the approximation is rather high. The maximal relative error does not excess 1% for any n value in the range $-0.5 \leqslant n \leqslant 5$.

Any other optimum parameter of the SPF and CPF with an insulated tip can be determined for the given n and an arbitrary geometrical or thermal parameter, using Table 1 to calculate N^*_{opt} and G^*_{opt} and Tables 1 and 2 from the first part of this study [11], where the relations between the given and required parameters as well parameter(s) N^*_{opt} or (and) G^*_{opt} are presented for the SPF and CPF, respectively.

3.3. Correction factor to the optimum characteristic of a fin with an insulated tip to determine the corresponding characteristic of this fin with a non-insulated tip

The specified values for fins with non-insulated tips are both the exponent n in the local heat transfer equation, fin tip heat loss ratio ω, and any one of the fin Biot numbers Bi_a, Bi, Bi_l for direct statement of a fin optimization problem and the thermal conductance $G_z \equiv g_b/(kz)$ of the SPF, or $G_c \equiv g_b(h_b/k^2)$ of the CPF for inverse formulation of this problem. As mentioned above, parameter ω and given Biot number or fin conductance, in general case denoted B, appear in all equations only in form of complex parameter, in general case denoted $B_\omega = \omega^p B$, where exponent p depends on what a fin parameter is given. Taking into account the comparatively week effect of the tip heat loss on the optimum characteristics of the fin, we present the required main optimum parameter Y_{opt} of a fin with a non-insulated tip as a product of

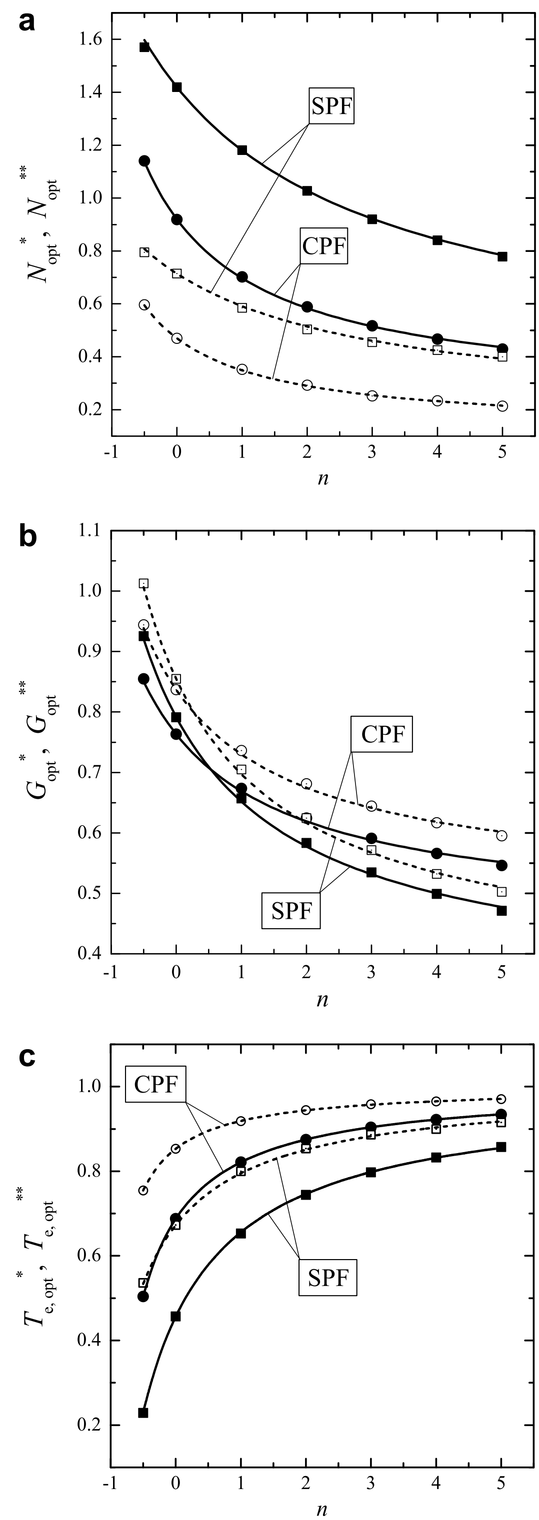

Fig. 5. Predicted optimum and inflection point thermo-geometrical parameters N^*_{opt} and N^{**}_{opt} (a), base conductances G^*_{opt} and G^{**}_{opt} (b) and tip temperatures $T^*_{e,opt}$ and $T^{**}_{e,opt}$ (c) as functions of the exponent n for the SPF and CPF with INT and NINT displayed by solid and dot-centered open squares and circles, respectively, and their approximations by the homographic function Eq. (16) with the coefficients collected in Table 1 (solid and short-dashed lines, respectively).

the corresponding characteristic Y^*_{opt} of this fin with an insulated tip which depends only on n and the correction

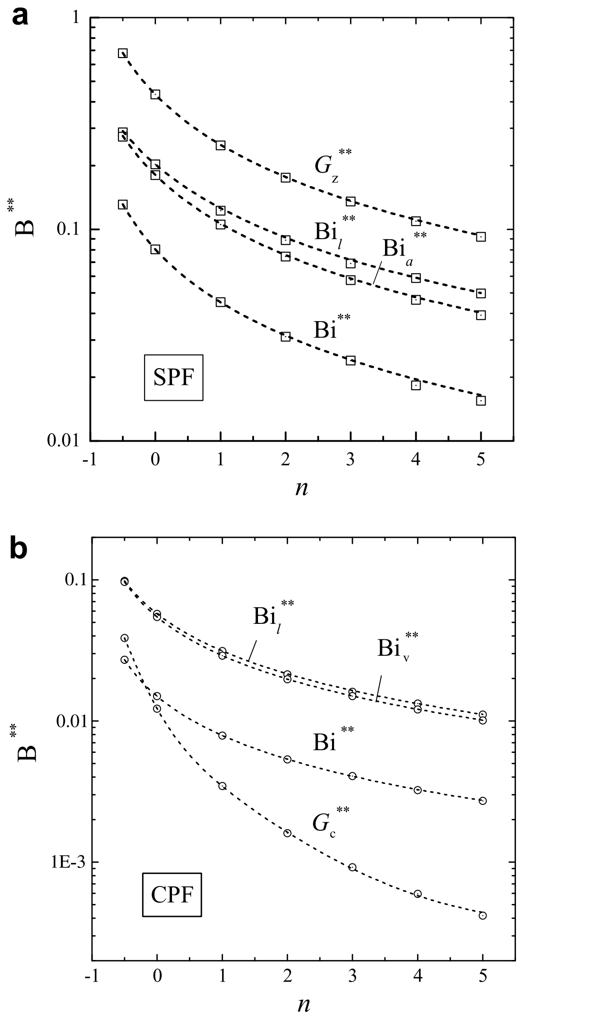

Fig. 6. Predicted maximum allowable B^{**} numbers for the SPF (a) and CPF (b) with a NINT as functions of the given exponent n (dot-centered open squares and circles, respectively) and their approximations by the homographic function Eq. (16) with the coefficients collected in Table 2 (short-dashed lines).

factor $\xi_{Y,B}$, which depends both on n and on the second complex parameter that, in general case, was designated above as B_ω

$$Y_{\rm opt} = \xi_{Y,B} Y^*_{\rm opt}, \qquad (17)$$

where $Y^*_{\rm opt}$ is calculated using Eq. (16) and Table 1.

Our analysis revealed that the correction factor $\xi_{Y,B}$ for any required main optimum value $Y_{\rm opt}$ (i.e. $G_{\rm opt}, T_{\rm e,opt}$ or $N_{\rm opt}$) and any given n, ω, and parameter B (i.e. any one of three fin Biot numbers or fin base conductance G_z or G_c for the SPF or CPF, respectively) can be well approximated by the following, general for all considered cases, function with respect to the general independent variable:

$$\beta_{\omega,B} = f(\omega)\cdot B/B^{**} = \omega^{\rm p}\cdot B/B^{**} = B/B^{**}_\omega, \qquad (18)$$

where

$$B^{**}_\omega = B^{**}/f(\omega) = B^{**}/\omega^{\rm p}. \qquad (19)$$

The relation that defines the dimensionless independent variable $\beta_{\omega,B}$ presented in Eq. (18) may be interpreted in two ways. On the one hand, $\beta_{\omega,B}$ is a ratio of given parameter B multiplied by a correction function $f(\omega)$ of given parameter ω to maximum allowable parameter B^{**} at $\omega = 1$. On the other hand, $\beta_{\omega,B}$ is a ratio of given parameter B to maximum allowable parameter B^{**}_ω at given ω. The latter parameter is defined by Eq. (19). The exponent p of power function $f(\omega) = \omega^{\rm p}$ for a fin of given form (SPF or CPF) depends on what a fin parameter B is specified. Its values are given in the right column of Table 2. The functions $f(\omega) = \omega^{\rm p}$ are presented in insets of Fig. 7a and d at given B equivalent to any of Biot numbers or G_z for the SPF and any Biot number or $G_{\rm c}$ for the CPF. The approximate expression for the correction factor to account for the fin tip heat loss can be presented as

$$\xi_{Y,B} \equiv Y_{\rm opt}/Y^*_{\rm opt} = 1 \pm \beta^{p_1}_{\omega,B}[\mu_Y - \nu_{Y,B}(1-\beta_{\omega,B})^{p_2}], \qquad (20)$$

where $0 \leqslant \beta_{\omega,B} \leqslant 1$. Exponents p_1 and p_2 are the numerical constants that differ for different B and Y. They are collected in Table 3 for both the SPF and CPF. Parameter $\mu_Y = (Y^{**}_{\rm opt}/Y^*_{\rm opt}) - 1$ for every main required parameter $Y_{\rm opt}$ depends only on n. And lastly, parameter $\nu_{Y,B}$ depends both on n and on the given as well as main required parameters B and $Y_{\rm opt}$. They are presented in Tables 4 and 5 for the SPF and CPF, respectively. The sign plus in the RHS of Eq. (20) refers to the cases of $Y_{\rm opt} = G_{\rm opt}$ and $Y_{\rm opt} = T_{\rm e,opt}$, whereas the sign minus refers to the case of $Y_{\rm opt} = N_{\rm opt}$.

Comparison between functions $\xi_{Y,B}$ vs $\beta_{\omega,B}$ for $Y \equiv N, G$, and $T_{\rm e}$ and $B \equiv Bi_{\rm a}$ (or $Bi_{\rm v}$), $Bi, Bi_{\rm l}$, and G_z (or $G_{\rm c}$) for the SPF (or CPF) presented in Fig. 7 shows that these functions calculated using approximate formula Eq. (20) and displayed by dot-centered open squares or circles are in good agreement with those calculated numerically and displayed by solid lines in the whole range of $\beta_{\omega,B}$ for all given and main required parameters.

Since the optimization problem is actually calculated only for given dimensionless Biot parameters $Bi_{\rm a}$ and $Bi_{\rm v}$ the corresponding optimum fin dimensions, specifically, its thickness or radius and height as well as corresponding Biot numbers Bi and $Bi_{\rm l}$ and fin thermal conductances G_z and $G_{\rm c}$ are, in essence, the dependent variables. However, in applications it is convenient to consider each of these dimensionless values as one of given parameters (other given parameters are n and ω), using the relations between dimensionless parameters collected in Tables 1 and 2 in the first part of this study [11]. Numerical calculations show that for given values of n, ω, and $Bi_{\rm l}$ the relation between the correction factor $\xi_{Y,Bi_{\rm l}}$ and $\beta_{\omega,Bi_{\rm l}}$ can be calculated not only in the range $0 \leqslant \beta_{\omega,Bi_{\rm l}} \leqslant 1$ but in range of $1 < \beta_{\omega,Bi_{\rm l}} \leqslant 1.07$ for the SPF and $1 < \beta_{\omega,Bi_{\rm l}} \leqslant 1.1$ for the CPF. As can be seen in Fig. 7, function $\xi_{Y,Bi_{\rm l}}$ vs $\beta_{\omega,Bi_{\rm l}}$ in this region is two-valued. But this region is physically impossible because as mentioned above, the optimization problem is meaning-

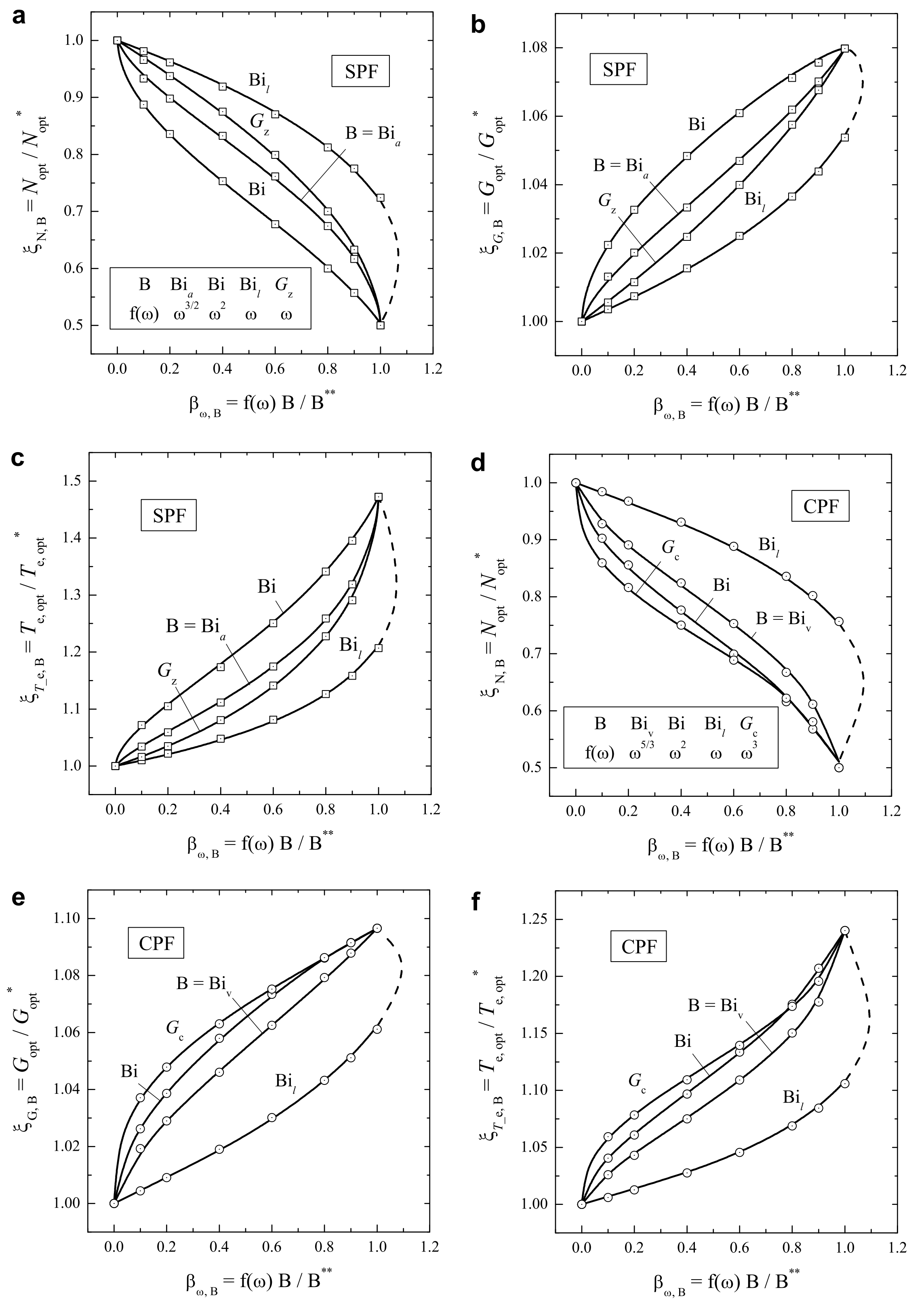

Fig. 7. Predicted correction factors $\xi_{N,B}$, $\xi_{G,B}$ and $\xi_{T_e,B}$ as functions of complex relative parameter $\beta_{\omega,B} = f(\omega)(B/B^{**})$ for the SPF (a–c) and CPF (d–f) displayed by solid lines. Functions $f(\omega)$ at different B given in inserts of the plots a and d are valid to plots a–c and d–f for the SPF and CPF, respectively. Calculations results using explicit approximation formula Eq. (20) are displayed by dot-centered open squares and circles for the SPF and CPF, respectively.

ful only for given parameter Bi_a (or G_z) for the SPF and Bi_v (or G_c) for the CPF, i.e. in the range $0 \leqslant \beta_{\omega,B} \leqslant 1$. Therefore, the curves ξ_{Y,Bi_l} vs β_{ω,Bi_l} for $\beta_{\omega,Bi_l} > 1$ are shown by dashed lines.

Table 3
Exponents p_1 and p_2 in Eq. (20) which are necessary to find the correction factor $\xi_{Y,B}$ and determine the required optimum parameter Y_{opt} of the SPF and CPF with non-insulated tips

Given parameter B	Required optimum parameter Y_{opt}	Fin type			
		SPF		CPF	
		p_1	p_2	p_1	p_2
Bi_a (or Bi_v)	N_{opt}	1/2	1/2	1/2	1/2
	G_{opt}	1/2	6/7	1/2	1
	$T_{e,opt}$	2/3	2/5	2/3	2/5
Bi	N_{opt}	1/2	2/3	1/2	2/3
	G_{opt}	1/2	5/2	1/2	5/2
	$T_{e,opt}$	1/4	3/4	1/2	5/8
Bi_l	N_{opt}	1	1/8	1	1/8
	G_{opt}	1	1/5	1	1/6
	$T_{e,opt}$	1	1/8	1	1/8
G_z (or G_c)	N_{opt}	3/4	1/2	1/3	1/2
	G_{opt}	1	3/4	1/3	1
	$T_{e,opt}$	1	1/3	1/3	1/2

Table 4
Coefficients C_i of the homographic function Eq. (16) intended for approximation of parameters μ_Y and $\nu_{Y,B}$ at given parameters n, ω, and B to find the required correction factor (c-factor) $\xi_{Y,B}$ and determine the main optimum parameters of the SPF with a non-insulated tip

Given parameter B	Required c-factor $\xi_{Y,B}$	Parameters in Eq. (20)	Coefficient C_i for i		
			1	2	3
$0 \leqslant B \leqslant B^{**}$	$\xi_{N,B}$	μ_N	0.5	0	0
	$\xi_{G,B}$	μ_G	0.0798	0.06206	0.9626
	$\xi_{T_e,B}$	μ_{T_e}	0.4722	0	1.1164
$0 \leqslant Bi_a \leqslant Bi_a^{**}$	ξ_{N,Bi_a}	ν_{N,Bi_a}	0.3036	0	0
	ξ_{G,Bi_a}	ν_{G,Bi_a}	0.04207	0.04078	1.1456
	ξ_{T_e,Bi_a}	ν_{T_e,Bi_a}	0.3273	0	1.1978
$0 \leqslant Bi \leqslant Bi^{**}$	$\xi_{N,Bi}$	$\nu_{N,Bi}$	0.1542	0	0
	$\xi_{G,Bi}$	$\nu_{G,Bi}$	0.01182	0.01137	1.2412
	$\xi_{T_e,Bi}$	$\nu_{T_e,Bi}$	0.3726	0	1.1715
$0 \leqslant Bi_l \leqslant Bi_l^{**}$	ξ_{N,Bi_l}	ν_{N,Bi_l}	0.3122	0	0
	ξ_{G,Bi_l}	ν_{G,Bi_l}	0.04435	0.0246	0.78401
	ξ_{T_e,Bi_l}	ν_{T_e,Bi_l}	0.3698	0	1.1626
$0 \leqslant G_z \leqslant G_z^{**}$	ξ_{N,G_z}	ν_{N,G_z}	0.3251	0	0
	ξ_{G,G_z}	ν_{G,G_z}	0.02643	0.02632	1.1554
	ξ_{T_e,G_z}	ν_{T_e,G_z}	0.3211	0	1.1955

4. Comparisons between our results and available literature data

Let us consider now several examples to compare results of our unified analysis of the fin optimization problems with the corresponding data in available literature.

4.1. Example 1: Comparison of maximum allowable Biot numbers with data by Yeh [9]

The functions of the maximum allowable Biot numbers $Bi_{a,\omega}^{**}$ and $Bi_{v,\omega}^{**}$ vs parameter ω in the range $0.15 \leqslant \omega \leqslant 4$ at given n for the SPF and CPF, respectively, are presented in Fig. 8a and b. Our results are determined by Eq. (19) in combination with Table 2 and plotted by solid straight lines using logarithmic scales. Data by Yeh [9] are shown by dot-centered open squares and circles for the SPF and CPF, respectively. The close agreement of our results with the data by Yeh presented in the range $0.15 \leqslant \omega \leqslant 1$ is quite evident.

Table 5
Coefficients C_i of the homographic function Eq. (16) intended for approximation of parameters μ_Y and $\nu_{Y,B}$ at given parameters n, ω, and B to find the required correction factor (c-factor) $\xi_{Y,B}$ and determine the main optimum parameters of the CPF with a non-insulated tip

Given parameter B	Required c-factor $\xi_{Y,B}$	Parameters in Eq. (20)	Coefficient C_i for i		
			1	2	3
$0 \leqslant B \leqslant B^{**}$	$\xi_{N,B}$	μ_N	0.5	0	0
	$\xi_{G,B}$	μ_G	0.09659	0.08789	0.99299
	$\xi_{T_e,B}$	μ_{T_e}	0.24013	0	1.03262
$0 \leqslant Bi_v \leqslant Bi_v^{**}$	ξ_{N,Bi_v}	ν_{N,Bi_v}	0.2866	0.21712	0.83195
	ξ_{G,Bi_v}	ν_{G,Bi_v}	0.03974	0.04956	1.29529
	ξ_{T_e,Bi_v}	ν_{T_e,Bi_v}	0.12501	0	1.11611
$0 \leqslant Bi \leqslant Bi^{**}$	$\xi_{N,Bi}$	$\nu_{N,Bi}$	0.20646	0.13608	0.79187
	$\xi_{G,Bi}$	$\nu_{G,Bi}$	0.01767	0	0
	$\xi_{T_e,Bi}$	$\nu_{T_e,Bi}$	0.11972	0	1.12936
$0 \leqslant Bi_l \leqslant Bi_l^{**}$	ξ_{N,Bi_l}	ν_{N,Bi_l}	0.34254	0	0
	ξ_{G,Bi_l}	ν_{G,Bi_l}	0.052	0	0
	ξ_{T_e,Bi_l}	ν_{T_e,Bi_l}	0.17902	0	1.05486
$0 \leqslant G_c \leqslant G_c^{**}$	ξ_{N,G_c}	ν_{N,G_c}	0.20748	0.15155	0.82519
	ξ_{G,G_c}	ν_{G,G_c}	0.01844	0	0
	ξ_{T_e,G_c}	ν_{T_e,G_c}	0.1184	0	1.10846

4.2. Example 2: Optimum aspect ratio and reduced thermal conductance of the CPF with an INT and NINT for given Bi_v, n, and ω

An exact relation between the required optimum aspect ratio $\alpha_{opt}^* = (l/d)_{opt}^*$ as well as reduced thermal conductance $\widehat{G}_{c,opt}^* = G_{c,opt}^*/(4\pi)$ of the CPF with an insulated tip and the given values of Bi_v, n, and ω is expressed by formulae given in Table 2 [11]:

$$\alpha_{opt}^* \equiv (l/d)_{opt}^* = [(4\pi N_{opt}^{*6})^{1/5}/4]Bi_v^{-3/5}, \tag{21}$$

$$\widehat{G}_{c,opt}^* = G_{opt}^*[Bi_v^3/(4\pi)]^{3/5}, \tag{22}$$

where the main optimum fin parameters N_{opt}^* and G_{opt}^* depend only on parameter n. These relations are approximated by the homographic function Eq. (16) with three numerical coefficients for every parameter given in Table 1:

$$N_{opt}^* = (0.9193 + 0.11223n)/(1 + 0.48075n), \tag{23}$$

$$G_{opt}^* = (0.76314 + 0.20122n)/(1 + 0.44145n). \tag{24}$$

Using Eq. (23) we calculate the factor $K_\alpha^* = (4\pi N_{opt}^{*6})^{1/5}/4$ in Eq. (21) and approximate it by the homographic function Eq. (16), which gives

$$K_\alpha^* = (0.37491 + 0.03509n)/(1 + 0.52297n). \tag{25}$$

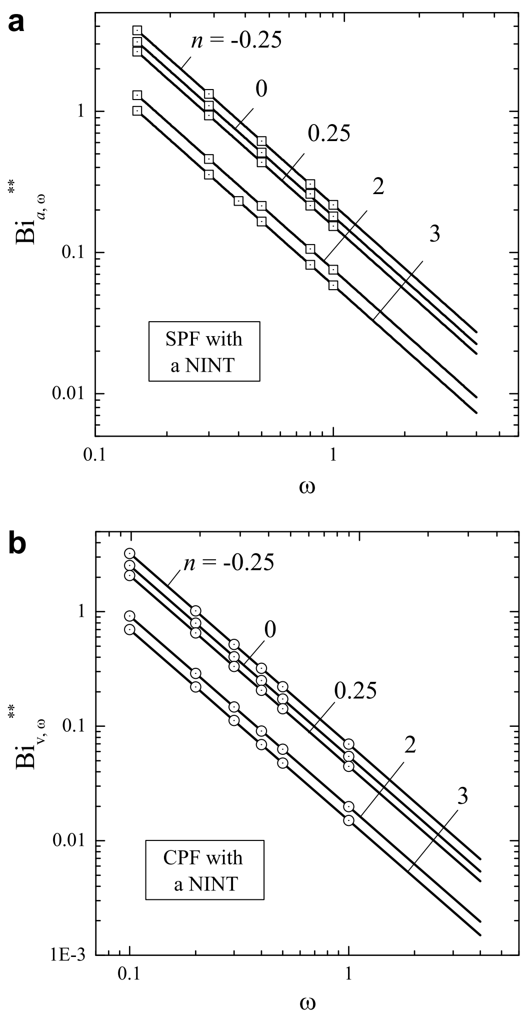

Fig. 8. Maximum allowable parameters $B^{**}_{a,\omega}$ and $B^{**}_{v,\omega}$ for the SPF (a) and CPF (b) with a NINT as functions of the ratio ω of heat transfer coefficients on the tip and lateral fin surfaces predicted by Eq. (16) with the coefficients collected in Table 2 (solid lines) and given in paper by Yeh [9] (dot-centered open squares and circles for the SPF and CPF, respectively).

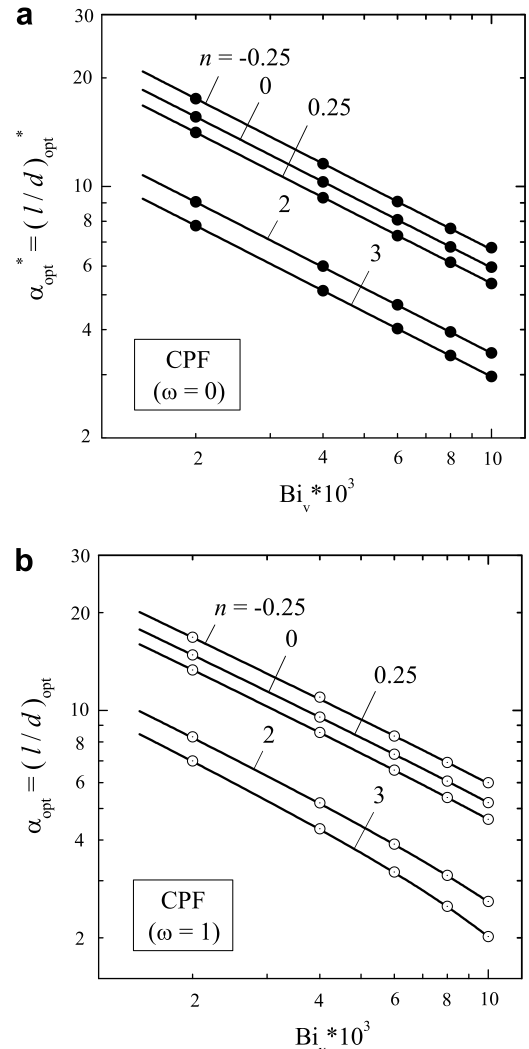

Fig. 9. Optimum aspect ratio α_{opt} as functions of the Biot number Bi_v for the CPF with an INT (a) and NINT at $\omega = 1$ (b) predicted by our method (solid lines) and by Yeh [9] (solid and dot-centered open circles, respectively).

It is seen from Eqs. (21) and (25) that the optimum aspect ratio of the CPF with an insulated tip can be presented as

$$\alpha^*_{opt} = K^*_\alpha Bi_v^{-3/5}. \tag{26}$$

Functions α^*_{opt} vs Bi_v calculated using Eqs. (25) and (26) for $n = -0.25$, 0, 0.25, 2 and 3 are shown in Fig. 9a by solid straight lines using logarithmic scale. It can be seen that these lines agree closely with the corresponding data of numerical calculations by Yeh [9] shown in the same plot by solid circles.

To determine N_{opt} and G_{opt} for the CPF with a non-insulated tip we use correction factors ξ_{N,Bi_v} and ξ_{G,Bi_v} which are defined in Eq. (20) with parameters given in Tables 2, 3, and 5. Then we calculate the main dimensionless parameters of the CPF with a non-insulated tip $N_{opt} = \xi_{N,Bi_v} N^*_{opt}$ and $G_{opt} = \xi_{G,Bi_v} G^*_{opt}$. After N_{opt} and G_{opt} are determined we use Eqs. (21) and (22) without asterisks to find functions α_{opt} vs Bi_v and $\widehat{G}_{c,opt}$ vs Bi_v at the same n values as above for the CPF with an insulated tip. Functions α_{opt} vs Bi_v plotted in Fig. 9b by solid lines using logarithmic scale for $\omega = 1$ and $n = -0.25$, 0, 0.25, 2 and 3 agree closely with the results of numerical calculations by Yeh [9] shown by dot-opened circles. Note that these curves are not the straight lines in logarithmic scale as distinct from the CPF with an insulated tip, especially for the high values of Bi_v and n.

The functions α_{opt} vs ω and $\widehat{G}_{c,opt} = G_{c,opt}/(4\pi)$ vs ω calculated using our method (solid lines) are compared in Fig. 10a and b with data by Yeh [9] (dot-centered open circles) for $Bi_v = 0.01$. It is seen that our results are in close agreement with the data by Yeh.

4.3. Example 3: Optimum aspect ratio and specific thermal conductance of the CPF with an insulated tip for given Bi_l and n

An exact relation between the required optimum aspect ratio $\alpha^*_{opt} = (l/d)_{opt}$ as well as specific thermal conductance

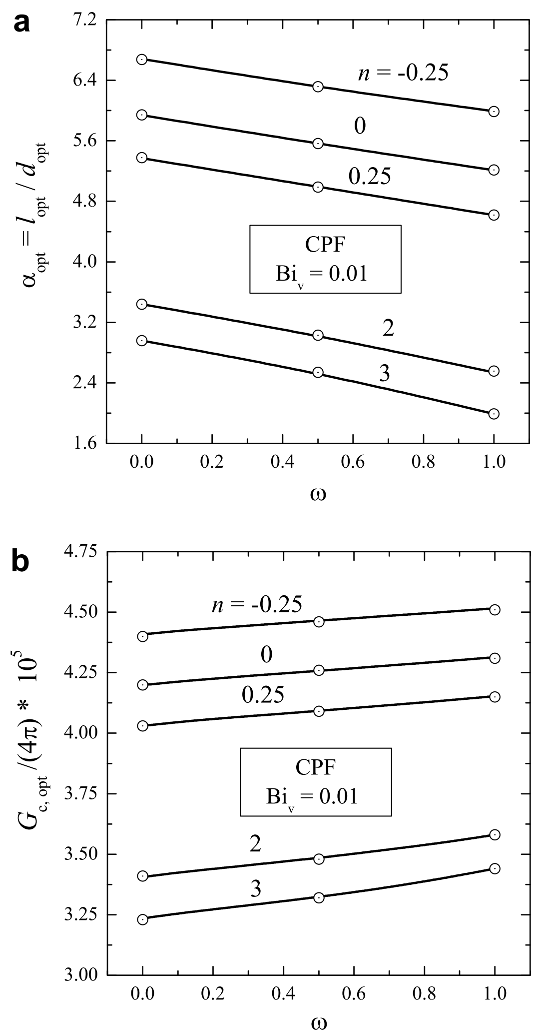

Fig. 10. Optimum aspect ratio α_{opt} (a) and generalized base thermal conductance G_c (b) of the CPF with a NINT at $Bi_v = 0.01$ as functions of the ratio ω of heat transfer coefficients on the tip and lateral fin surfaces predicted by our method (solid lines) and by Yeh [9] (dot-centered open circles).

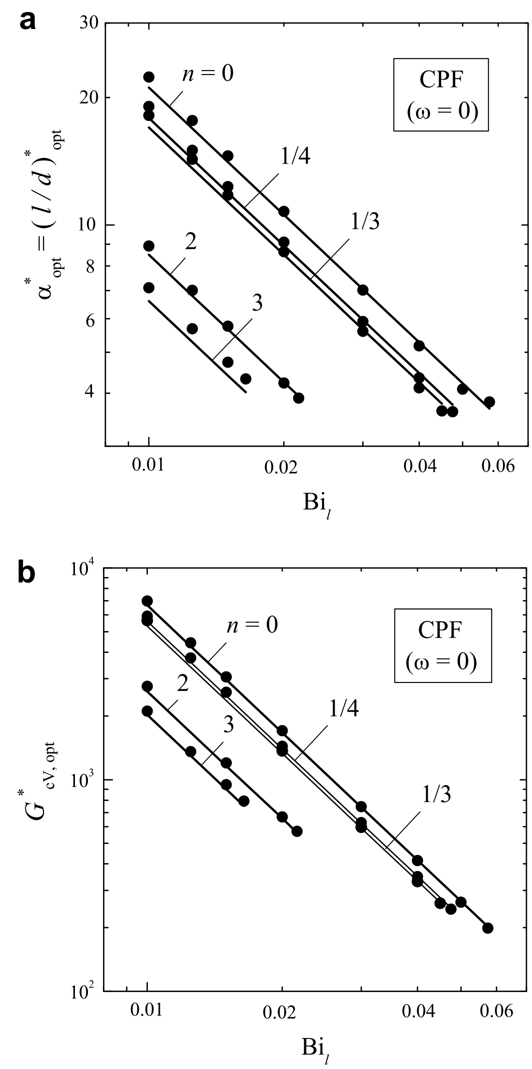

Fig. 11. Optimum aspect ratio α_{opt} (a) and dimensionless fin thermal conductance per unit volume G^*_{cVopt} (b) of the CPF with an INT predicted by our method (solid lines) and by K. Laor and H. Kalman [11].

$G^*_{cV,opt} = G^*_{c,opt}/V_{opt}$ of the CPF with an insulated tip and the given values of Bi_l and n is expressed by formulae collected in Table 2 [11]

$$\alpha^*_{opt} \equiv (l/d)^*_{opt} = [(N^*_{opt})^2/4]/Bi_l, \tag{27}$$

$$G^*_{cV,opt} = G^*_{opt}(N^*_{opt})^{8/5}/Bi_l^2. \tag{28}$$

Corresponding relations numerically calculated in paper [12] by Laor and Kalman are approximated by following power functions which in our denotations expressed as

$$\alpha^*_{opt} \equiv (l/d)^*_{opt} = K_2/Bi_l^{H_2}, \tag{29}$$

$$G^*_{cV,opt} = K_1/Bi_l^{-H_1}, \tag{30}$$

where K_1 and K_2 as well as H_1 and H_2 are numerical constants depending only on n and given in Table 3 of the paper [12]. Functions α^*_{opt} vs Bi_l and $G^*_{cV,opt}$ vs Bi_l calculated using Eqs. (29) and (30) for $n = -0.25$, 0, 0.25, 2 and 3 are shown in Fig. 11a and b by solid straight lines using logarithmic scales. It can be seen that these lines agree reasonably well with the corresponding data in the paper [12] by Laor and Kalman shown in the same plots by solid circles. The case $n = 3$ in Fig. 11a is one exception when our straight line lies noticeably below than data in paper [12]. Notice that Eqs. (27) and (28) are exact ones. Thus, comparing Eqs. (29) and (30) with Eqs. (27) and (28) one can conclude that exponents H_2 and H_1 must be equal 1 and -2, respectively, for any value of n but not for so-called "ideal" fin only. In paper [12] the ideal fin was defined as a fin with an insulated tip and uniform heat transfer coefficient ($n = 0$). If one assumes that $H_2 = 1$ and $H_1 = -2$ for any value of n then K_2 must be equal to $(N^*_{opt})^2/4$ and K_1 equal to $G^*_{opt}(N^*_{opt})^{8/5}$.

For the fins with a non-insulated tips the above parameters depend not only on n and Bi_l but in addition on given value of ω to account for the heat transfer from the fin tip. Our method intended for determination of the correction factors to $N^*_{\rm opt}$ and $G^*_{\rm opt}$ to find $N_{\rm opt}$ and $G_{\rm opt}$ for given values of $Bi_{\rm l}$, n and ω is analogous to one presented above for given values of $Bi_{\rm v}$, n, and ω.

4.4. Example 4: Correction factors ξ_{N,G_z} and ξ_{G,G_z} for the optimum SPF with given values of G_z, n and ω

As it is mentioned above, our method can be successfully used to optimize the fin dimensions and to find all optimum fin characteristics not only for direct but also for inverse statement of a heat transfer problem. In this case fin base thermal conductance G_z, n, and ω are given, whereas correction factors ξ_{N,G_z} and ξ_{G,G_z} have to be determined. Such optimization problem has been solved by Razelos and Krikkis in [6] for the SPF with insulated and non-insulated tips and uniform heat transfer coefficient ($n=0$) for different values of the tip heat transfer ratio ω.

Now we are going to solve this problem using our general approach developed in present paper and compare our results with data by Razelos and Krikkis [6]. To determine $N_{\rm opt}$ and $G_{\rm opt}$ for the SPF with a non-insulated tip we use correction factors ξ_{N,G_z} and ξ_{G,G_z} which are defined in Eq. (20) with parameters given in Tables 2–4:

$$\xi_{N,G_z} = 1 - \beta^{3/4}_{\omega,G_z}[\mu_N - \nu_{N,G_z}(1-\beta_{\omega,G_z})^{1/2}], \quad (31)$$

$$\xi_{G,G_z} = 1 + \beta_{\omega,G_z}[\mu_G - \nu_{G,G_z}(1-\beta_{\omega,G_z})^{3/4}], \quad (32)$$

where according to Eq. (18) and Tables 2 and 3

$$\beta_{\omega,G_z} = \omega G_z / G_z^{**}, \quad (33)$$

$$G_z^{**} = 0.43297/(1+0.72642n). \quad (34)$$

It follows also from Table 4 that $\mu_N = 0.5$, $\nu_{N,G_z} = 0.3251$,

$$\mu_G = (0.0798 + 0.06206n)/(1+0.9626n), \quad (35)$$

$$\nu_{G,G_z} = (0.02643 + 0.02632n)/(1+1.1554n) \quad (36)$$

and for given $n=0$ parameter $G_z^{**} = 0.43297$, $\mu_G = 0.0798$, and $\nu_{G,G_z} = 0.02643$.

Razelos and Krikkis have approximated relations between the optimum parameters of the SPF with a non-insulated tip $N_{\rm opt}$, $D_{\rm opt}$ and given values $\widehat{G}_{z,\omega}$ (in our denotations) by polynomial expressions $N_{\rm opt} = \sum_{m=0}^{3} a_m \widehat{G}^m_{z,\omega}$, $D_{\rm opt} = \sum_{m=0}^{3} b_m \widehat{G}^m_{z,\omega}$, where coefficients a_0–a_3 and b_0–b_3 are given in Table 1 of paper [6]. Taking into account that $\xi_{N,G_z} \equiv N_{\rm opt}/N^*_{\rm opt} = \left(\sum_{m=0}^{3} a_m \widehat{G}^m_{z,\omega}\right)/a_0$, $G_{\rm opt} = D_{\rm opt}/N^{1/3}_{\rm opt}$, and $\xi_{G,G_z} \equiv G_{\rm opt}/G^*_{\rm opt} = \left(\sum_{m=0}^{3} b_m \widehat{G}^m_{z,\omega}\right)/b_0$ one can transform relations $N_{\rm opt}$ and $D_{\rm opt}$ against $\widehat{G}_{z,\omega}$ into relations ξ_{N,G_z} and ξ_{G,G_z} against β_{ω,G_z}.

Functions ξ_{N,G_z} and ξ_{G,G_z} against β_{ω,G_z} obtained using Eqs. (31) and (32) (solid lines) are compared in Fig. 12 with results obtained using above presented transformation of correlations by Razelos and Krikkis [6] (dot-centered squares). It can be seen that there is a close agreement between our results and data [6].

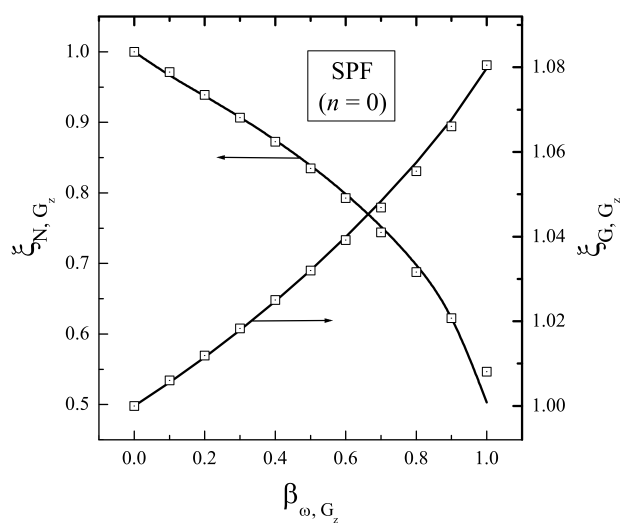

Fig. 12. Correction factors ξ_{N,G_z} and ξ_{G,G_z} of the SPF with a NINT as functions of complex relative parameter β_{ω,G_z} calculated by Eqs. (31) and (32) (solid lines) and determined according to correlations by Razelos and Krikkis [6] (dot-centered open squares).

4.5. Example 5: Radiating fins

Dimensionless relations between the optimum geometrical dimensions of the SPF (or CPF) with an insulated tip transfer heat only by radiation into surroundings with a zero absolute temperature for given base thermal

Table 6
Comparison of numerical factors (NF) in dimensionless relations between the optimum parameters of the SPF (or CPF) with insulated tips which transfer heat by radiation only into surroundings with zero absolute temperature ($n=3$) and given fin thermal conductance G_z (or G_c) determined according to present study with data by Wilkins [13]

Fin type	Relation between the required optimum fin parameter and given fin thermal conductance	Numerical factor (NF)		φ, %
		Present study	[13]	
SPF	$2Bi^*_{\rm opt} = [1/(2N^{*2/3}_{\rm opt}G^{*2}_{\rm opt})]G_z^2 = {\rm NF}\cdot G_z^2$	1.8508	1.8484	0.14
	$Bi^*_{\rm l,opt} = [N^{*2/3}_{\rm opt}/(2G^*_{\rm opt})]G_z = {\rm NF}\cdot G_z$	0.8872	0.8844	0.28
	$A^*_{\rm p,opt} = [1/(4G^{*3}_{\rm opt})]G_z^3 = {\rm NF}\cdot G_z^3$	1.6421	1.6347	0.47
CPF	$2Bi^*_{\rm opt} = 2/[N^{*2/5}_{\rm opt}(4\pi G^*_{\rm opt})^{2/3}]G_c^{2/3} = {\rm NF}\cdot G_c^{2/3}$	0.6878	0.6870	0.12
	$Bi^*_{\rm l,opt} = [N^{*4/5}_{\rm opt}/(4\pi G^*_{\rm opt})^{1/3}]G_c = {\rm NF}\cdot G_c$	0.3016	0.2999	0.57
	$V^*_{\rm opt} = [4\pi/(4\pi G^*_{\rm opt})^{1/3}]G_c^{5/3} = {\rm NF}\cdot G_c^{5/3}$	0.4482	0.4447	0.79

conductance G_z (or G_c) are used to determine corresponding numerical factors (NF) and to compare its values with data by Wilkins [13]. The obtained relations and values of NF are collected in Table 6. It is seen that NF like the main dimensionless optimum parameters of a fin with an insulated tip N^*_{opt} and G^*_{opt} depends only on exponent n since it is expressed only in terms of these parameters. Therefore, firstly these parameters are determined for given $n = 3$ using Eq. (16) and Table 1: $N^*_{opt} = 0.92226$ and $G^*_{opt} = 0.53397$ for the SPF as well as $N^*_{opt} = 0.51428$ and $G^*_{opt} = 0.58804$ for the CPF. Then every NF is determined using expression given in second column of Table 6. It is seen that our values of NF agree very well with ones found by Wilkins [13]. Maximum relative discrepancies do not exceed 0.14–0.47% for the SPF and 0.12–0.79% for the CPF.

5. Conclusions

(1) The relations between the main dimensionless fin parameters, specifically, G vs T_e and N vs T_e in the whole range of T_e ($0 \leqslant T_e \leqslant 1$) are calculated using an exact hypergeometric solution of the 1D steady-state heat transfer equation for a single SPF and CPF with an insulated and non-insulated tip. The local heat transfer coefficient is assumed to vary as a power function of the local excess temperature with an arbitrary value of exponent n in the range of $-0.5 \leqslant n \leqslant 5$.

(2) Every curve from G vs T_e set for a fin with an insulated tip is found to have $G^* = 0$ both at $T_e = 0$ and at $T_e = 1$ as well as a single global maximum $G^* = G^*_{opt}$ which occurs at $T_e = T^*_{e,opt}$. The thermo-geometrical fin parameter $N = N^*_{opt}$ from the N vs T_e set corresponds to optimum value of $T_e = T^*_{e,opt}$. Therefore, the main and all other optimum parameters for a fin with an insulated tip depend only on exponent n.

(3) Every curve from G vs T_e set for a fin with a non-insulated tip is found to depend not only on n but also on the additional complex parameter $Bi_{a,\omega} = \omega^p Bi_a$ for the SPF (or $Bi_{v,\omega} = \omega^p Bi_v$ for the CPF) which is a product of two given parameters, i.e. Biot number based on given profile area of the SPF (or volume of the CPF) and a power function ω^p of the given tip loss ratio ω. The exponent p of this function for a fin of given form depends on the given fin parameter in general case denoted by B. The value of p is equal to 3/2, 2, 1, and 1 at $B \equiv Bi_a, Bi, Bi_l$, and G_z for the SPF as well as p is equal to 5/3, 2, 1, and 3 at $B \equiv Bi_v, Bi, Bi_l$, and G_c for the CPF, respectively.

(4) Every curve from G vs T_e set for a fin with a non-insulated tip are shown to have $G = 0$ at $T_e = 0$, $G = \infty$ at $T_e = 1$ as well as a single local maximum and local minimum points in the range $0 < Bi_{a,\omega} < Bi^{**}_a$ for the SPF or $0 < Bi_{v,\omega} < Bi^{**}_v$ for the CPF. As $Bi_{a,\omega}$ (or $Bi_{v,\omega}$) increases these points approach each other. If $Bi_{a,\omega}$ (or $Bi_{v,\omega}$) reaches the value Bi^{**}_a (or Bi^{**}_v), then maximum and minimum G values merge and corresponding curve G vs T_e has the only inflection point. At $Bi_{a,\omega} > Bi^{**}_a$ (or $Bi_{v,\omega} > Bi^{**}_v$) the curves G vs T_e have no maximum and optimization problem loses its meaning. Parameters N^{**}_{opt}, G^{**}_{opt} and $T^{**}_{e,opt}$ for the inflection point as well as maximum allowable B^{**} values corresponding to the curve passing through this point are found to depend only on n.

(5) The main optimum parameters of a fin with an insulated tip N^*_{opt}, G^*_{opt}, and $T^*_{e,opt}$ as well as inflection point parameters of a fin with a non-insulated tip N^{**}_{opt}, G^{**}_{opt}, $T^{**}_{e,opt}$ and maximum allowable B^{**} values are approximated by general homographic function with three numerical coefficients for each parameter. These coefficients are collected in Tables 1 and 2.

(6) Each main optimum parameter of a fin with a non-insulated tip is expressed as a product of the corresponding parameter for this fin with an insulated tip and a correction factor approximated by explicit closed-form formula, which proved to be general for all required and given parameters. Correction factor is determined using Eq. (20) as function of complex relative fin tip parameter $\beta_{\omega,B} = \omega^p B/B^{**}$, for given value of n. Parameter B^{**}, exponents p_1 and p_2 as well as coefficients μ_Y and $\nu_{Y,B}$ as functions of n, given parameter B and required parameter Y are collected in Tables 2–5.

(7) Closed-form explicit relations for the optimum characteristics of the SPF and CPF for different combinations of given and required parameters obtained in this study were verified by comparison with large body of expressions, plots and calculation data available in numerous papers and books. These comparisons and given numerical examples confirm the validity and high accuracy of the obtained relations. It is believed that this study make clear and explicit relations between the dimensionless variables, which are of prime importance to a thermal design and optimization of single fins. It gives a simple and high-accuracy tool that can be successfully used in further theoretical studies and optimizations of single fins with non-uniform heat transfer coefficients, which depend on the fin temperature excess.

References

[1] A.D. Kraus, A. Aziz, J. Welty, Extended Surface Heat Transfer, Wiley, New York, 2001.

[2] A. Aziz, Optimum dimensions of extended surfaces operating in convective environment, Appl. Mech. Rev. 45 (5) (1992) 155–173.

[3] A. Sonn, A. Bar-Cohen, Optimum cylindrical pin fin, Trans. ASME, J. Heat Transfer 103 (4) (1981) 814–815.

[4] P. Razelos, The optimum dimensions of convective pin fins, Trans. ASME, J. Heat Transfer 105 (2) (1983) 411–413.

[5] P. Razelos, The optimum dimensions of convective pin fins with internal heat generation, J. Franklin Inst. 321 (1) (1986) 1–19.

[6] P. Razelos, R.N. Krikkis, On the optimum thermal design of individual longitudinal with rectangular profile, Int. Commun. Heat Mass Transfer 30 (3) (2003) 349–358.

[7] B.T.F. Chung, J.R. Iyer, Optimum design of longitudinal rectangular fins and cylindrical spines with variable heat transfer coefficient, Heat Transfer Eng. 14 (1) (1993) 31–42.
[8] R.H. Yeh, S.P. Liaw, An exact solution for thermal characteristics of fins with power-law heat transfer coefficient, Int. Commun. Heat Mass Transfer 17 (3) (1990) 317–330.
[9] R.H. Yeh, An analytical study of the optimum dimensions of rectangular fins and cylindrical pin fins, Int. J. Heat Mass Transfer 40 (15) (1997) 3607–3615.
[10] R.H. Yeh, Errors in one-dimensional fin optimization problem for convective heat transfer, Int. J. Heat Mass Transfer 39 (14) (1996) 3607–3615.
[11] I.N. Dul'kin, G.I. Garas'ko, Analysis of the 1D heat conduction problem for a single fin with temperature dependent heat transfer coefficient – I. Extended inverse and direct solutions, Int. J. Heat Mass Transfer, in this issue, doi:10.1016/j.ijheatmasstransfer.2007.11.027.
[12] K. Laor, H. Kalman, Performance and optimum dimensions of different cooling fins with a temperature-dependent heat transfer coefficient, Int. J. Heat Mass Transfer 39 (9) (1996) 1993–2003.
[13] J.E. Wilkins Jr., Minimum-mass thin fins which transfer heat only by radiation to surroundings at absolute zero, J. Soc. Ind. Appl. Math. 8 (4) (1960) 630–639.